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Stable Equilibrium Based on Lévy Statistics: A Linear Boltzmann Equation Approach

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To obtain further insight on possible power law generalizations of Boltzmann equilibrium concepts, we consider stochastic collision models. The models are a generalization of the Rayleigh collision model, for a heavy one dimensional particle M interacting with ideal gas particles with a mass $m \ll M$. Similar to previous approaches we assume elastic, uncorrelated, and impulsive collisions. We let the bath particle velocity distribution function to be of general form, namely we do not postulate a specific form of power-law equilibrium. We show, under certain conditions, that the velocity distribution function of the heavy particle is Lévy stable, the Maxwellian distribution being a special case. We demonstrate our results with numerical examples. The relation of the power law equilibrium obtained here to thermodynamics is discussed. In particular we compare between two models: a thermodynamic and an energy scaling approaches. These models yield insight into questions like the meaning of temperature for power law equilibrium, and into the issue of the universality of the equilibrium (i.e., is the width of the generalized Maxwellian distribution functions obtained here, independent of coupling constant to the bath).

KEY WORDS: Lévy statistics; collision models; generalized equilibrium.

1. INTRODUCTION

Khintchine[1] revealed the relation between the Gaussian Central Limit Theorem and the Boltzmann–Gibbs statistics. From a stochastic point of view, we may see this relation by considering the velocity of a Brownian particle. The phenomenological dynamical description of a Brownian motion, is in terms of the Langevin equation $\dot{V}_M = -\gamma V_M + \eta(t)$, where $\eta(t)$ is a *Gaussian* white noise term. Usually the assumption of Gaussian noise is

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imposed on the Langevin equation to obtain a Maxwellian velocity distribution describing the equilibrium of the Brownian particle. We may reverse our thinking of the problem, the Gaussian noise is naturally expected based on Central Limit Theorem arguments, and the latter leads to Maxwell's equilibrium. Similar arguments hold for a Brownian particle in an external time independent binding force field, the Maxwell–Boltzmann equilibrium is obtained in the long time limit only if the noise term is Gaussian.

However, the Gaussian Central Limit Theorem is nonunique. Lévy and Khintchine[(2)] have generalized the Gaussian Central Limit Theorem, to the case of summation of independent, identically distributed random variables described by long tailed distributions. In this case Lévy distributions replace the Gaussian in generalized limit theorems. Hence it is natural to ask[(3–6)] if a Lévy based statistical mechanics exist? And if so what is its physical domain and its relation to thermodynamics. This type of questions are timely due to the interest in power law generalizations of statistical mechanics.[(7–11)] While Lévy statistics is used in many applications,[(12–18)] its *possible* relation to a generalized form of equilibrium statistical mechanics is still unclear.

As well known, Boltzmann used a kinetic approach for a dilute gas of particles to derive the Maxwell velocity distribution (see details in ref. 19). Thus, the kinetic approach can be used as a tool to derive equilibrium, starting from nonequilibrium dynamics. For example, Ernst[(20)] and Chernov and Lebowitz[(21)] show how the Maxwell equilibrium is obtained from different types of collision models. Thus, the Maxwell equilibrium transcends details of individual kinetic models. It is important to note that in these works, important boundary and initial conditions are imposed on the dynamics.[(19–23)] For example, the second moment of the velocity of the particles is supposed to be finite, Eqs. (1.7) and (2.10) in ref. 20. A possible domain of power law generalizations of Maxwell's equilibrium, are the cases where one does not impose "finite variance" initial and/or boundary conditions.

A step beyond the Maxwellian velocity distribution was obtained in the kinetic theory of inelastic systems. Extensive investigations,[(24–29)] using the inelastic Maxwell model, or inelastic models for tracer diffusion,[(30)] shows that initial Gaussian distributions of velocities are driven into power law or stretched exponential distributions (however no direct relation to Lévy statistics was obtained). Bobylev and Cercignani,[(31)] investigated a nonlinear Boltzmann equation with an infinite velocity variance showing that the solution exists, and obtaining certain bounds on it. In ref. 31 the *possibility* of a relation between solutions of the Boltzmann equation and the Lévy Central Limit Theorem was briefly pointed out. In ref. 32 I investigated a driven Maxwell model, using a nonlinear Boltzmann equation, showing that steady state solutions of the model are Lévy stable. After this manuscript was submitted, two additional publications emphasized the relation between steady state solutions of kinetic models and Lévy statistics. Pulvirenti and Toscani[33] showed that equilibrium in an inelastic Kac model, under certain conditions yields a Lévy stable law. Zanette and Montemurro[34] investigated thermal measurements of stationary nonequilibrium power law systems, showing that Lévy statistics describes fluctuations of a model thermostat. We note that Lévy distribution of velocities of vortex elements was observed in turbulent flows and also from numerical simulations by Min *et al.*[35] (see also refs. 36–39). In this case the mechanism leading to Lévy statistics are long range interactions.

In this manuscript we will consider a simple kinetic approach to obtain a generalization of Maxwell's velocity distribution. Briefly we consider a one dimensional tracer particle of mass M randomly colliding with gas particles of mass $m \ll M$. Two main assumptions are used: (i) molecular chaos holds, implying lack of correlations in the collision process (Stoszzahlansantz), and (ii) rate of collisions is independent of the energy of the colliding particles, i.e., our model belongs to the family of Maxwell models. Let the probability density function (PDF) of velocity of the gas particles be $f(\tilde{v}_m)$. If $f(\tilde{v}_m)$ is Maxwellian, the model describes Brownian type of motion for the heavy particle, the equilibrium being the Maxwell distribution.[40–42] Since our aim is to investigate generalized equilibrium, we do not impose the standard condition of Maxwellian velocity distribution for the bath particles. Our aim is to show, under general conditions, that the equilibrium PDF of the tracer particle $W(V_M)$ is stable. Stable means that wide classes of bath particles velocity PDFs $f(\tilde{v}_m)$, yield a unique equilibrium for the tracer particle $W_{\text{eq}}(V_M)$. Namely the equilibrium velocity distribution $W_{\text{eq}}(V_M)$ transcends the details of precise shape of $f(\tilde{v}_m)$. One of those stable velocity distributions will turn out to be the Maxwell velocity distribution.

To obtain an equilibrium we use two approaches which we call the energy scaling approach and the thermodynamic approach. The energy scaling approach is based on the assumption that an energy scale controls the velocity distribution of the bath particles. This means that we assume

$$f(\tilde{v}_m) = \frac{1}{\sqrt{T/m}} q(\tilde{v}_m/\sqrt{T/m}), \tag{1}$$

and $q(x)$ is a nonnegative normalized function. Here T has similar meaning as the usual temperature. From this general starting point stable equilibrium is derived for the tracer particle.

In the thermodynamic approach we use a different argument. As well known the equilibrium velocity distribution of a Brownian particle M immersed in a fluid, is independent of the mass of the the fluid particles m. Thus the heavy Brownian particle reaches an equilibrium which does not depend on the mass of atoms or molecules in its surroundings. In the thermodynamic approach we will impose a similar behavior on the tracer particle M, imposing the condition that its equilibrium be independent of m. This approach leads us to a generalized temperature concept. The main motivation for this approach is to check if it makes sense to assume that power law generalized equilibrium for the heavy particle are independent from its coupling to the bath. Namely can we expect a universal behavior for the power-law velocity distributions which are not sensitive to the details of the model and to the interaction of tracer particle-bath particle.

It is shown that the two approaches yield stable Lévy equilibrium. However, besides the Maxwellian case the two approaches yield different types of equilibrium. Thus, according to our model generalized equilibrium based on Lévy statistics naturally emerges, however this type of equilibrium is different from the standard equilibrium. One cannot generally treat the tracer particle equilibrium properties as separable from coupling to the bath, and at the same time use standard temperature concept T. Namely, Lévy velocity distribution are indeed the natural generalization of the Maxwellian velocity distribution, however the width of the Lévy distribution is not related in a simple way to temperature as found for the Maxwellian case.

This manuscript is organizes as follows. In Section 2 I present the model, and the linear Boltzmann equation under investigation. The time dependent solution of the model is found in Fourier space and an exact solution of the problem is found for a special case. In Section 3 the equilibrium solution of the Boltzmann equation is obtained, this solution is valid for any mass ratio $\epsilon \equiv m/M$. In Section 4 the Maxwell Velocity distribution is derived. The thermodynamic approach is considered in Section 5 and the energy scaling approach in Section 6. Throughout this work numerically exact solutions of the model are compared with asymptotic solution obtained in the limit of weak collisions $\epsilon \to 0$. We end with a brief summary. A very brief summary of part of our results appeared in ref. 32.

2. MODEL AND TIME DEPENDENT SOLUTION

We consider a one dimensional tracer particle with the mass M coupled with bath particles of mass m. The tracer particle velocity is V_M. At random times the tracer particle collides with bath particles whose velocity is denoted with $\tilde{v}_m$. Collisions are elastic hence from conservation of momentum and energy

$$V_M^+ = \xi_1 V_M^- + \xi_2 \tilde{v}_m, \tag{2}$$

where

$$\xi_1 = \frac{1-\epsilon}{1+\epsilon} \qquad \xi_2 = \frac{2\epsilon}{1+\epsilon} \tag{3}$$

and $\epsilon \equiv m/M$ is the mass ratio. In Eq. (2) V_M^+ (V_M^-) is the velocity of the tracer particle after (before) a collision event. The duration of the collision events is much shorter than any other time scale in the problem. The collisions occur at a uniform rate R independent of the velocities of colliding particles. The probability density function (PDF) of the bath particle velocity is $f(\tilde{v}_m)$. This PDF does not change during the collision process, indicating that re-collisions of the bath particles and the tracer particle are neglected.

We now consider the equation of motion for the tracer particle velocity PDF $W(V_M, t)$ with initial conditions concentrated on $V_M(0)$. Standard kinetic considerations yield

$$\frac{\partial W(V_M, t)}{\partial t} = -RW(V_M, t) + R \int_{-\infty}^{\infty} \mathrm{d}V_M^- \int_{-\infty}^{\infty} \mathrm{d}\tilde{v}_m \, W(V_M^-, t) \, f(\tilde{v}_m) \times \delta(V_M - \xi_1 V_M^- - \xi_2 \tilde{v}_m), \tag{4}$$

where the delta function gives the constrain on energy and momentum conservation in collision events. Us-usual the first (second) term, on the right hand side of Eq. (4), describes a tracer particle leaving (entering) the velocity point V_M at time t. Equation (4) yields the forward master equation, also called the linear Boltzmann equation

$$\frac{\partial W(V_M, t)}{\partial t} = -RW(V_M, t) + \frac{R}{\xi_1} \int_{-\infty}^{\infty} \mathrm{d}\tilde{v}_m \, W\left(\frac{V_M - \xi_2 \tilde{v}_m}{\xi_1}\right) f(\tilde{v}_m). \tag{5}$$

This equation is valid for $\xi_1 \neq 0$ namely $\epsilon \neq 1$. In Eq. (5) the second term on the right hand side is a convolution in the velocity variables, hence we will consider the problem in Fourier space. Let $\bar{W}(k, t)$ be the Fourier transform of the velocity PDF

$$\bar{W}(k, t) = \int_{-\infty}^{\infty} W(V_M, t) \exp(ikV_M) \, \mathrm{d}V_M, \tag{6}$$

we call $\bar{W}(k, t)$ the tracer particle characteristic function. Using Eq. (5), the equation of motion for $\bar{W}(k, t)$ is a finite difference equation

$$\frac{\partial \bar{W}(k, t)}{\partial t} = -R\bar{W}(k, t) + R\bar{W}(k\xi_1, t)\, \bar{f}(k\xi_2), \tag{7}$$

where $\bar{f}(k)$ is the Fourier transform of $f(\tilde{v}_m)$. In Appendix A the solution of the equation of motion Eq. (7) is obtained by iterations

$$\bar{W}(k, t) = \sum_{n=0}^{\infty} \frac{(Rt)^n \exp(-Rt)}{n!} e^{ikV_M(0)\, \xi_1^n} \Pi_{i=1}^{n} \bar{f}(k\xi_1^{n-i}\xi_2), \tag{8}$$

with the initial condition $\bar{W}(k, 0) = \exp[ikV_M(0)]$. Similar analysis for the case $\xi_1 = 0$ shows that Eq. (8) is still valid with $\bar{f}(k\xi_1^{n-i}\xi_2) = \bar{f}(k)\, \delta_{ni}$ and $\xi_1^n = \delta_{n0}$ where δ_{ni} is the Kronecker delta.

The solution Eq. (8) has a simple interpretation. The probability that the tracer particle has collided n times with the bath particles is given according to the Poisson law

$$P_n(t) = \frac{(Rt)^n}{n!} \exp(-Rt), \tag{9}$$

reflecting the assumption of uniform collision rate. Let $W_n(V_M)$ be the PDF of the tracer particle conditioned that the particle experiences n collision events. It can be shown that the Fourier transform of $W_n(V_M)$ is

$$\bar{W}_n(k) = e^{ikV_M(0)\, \xi_1^n} \Pi_{i=1}^{n} \bar{f}(k\xi_1^{n-i}\xi_2). \tag{10}$$

Thus Eq. (8) is a sum over the probability of having n collision events in time interval $(0, t)$ times the Fourier transform of the velocity PDF after exactly n collision event

$$\bar{W}(k, t) = \sum_{n=0}^{\infty} P_n(t)\, \bar{W}_n(k). \tag{11}$$

It follows immediately that the solution of the problem is

$$W(V_M, t) = \sum_{n=0}^{\infty} P_n(t)\, W_n(V_M), \tag{12}$$

where $W_n(V_M)$ is the inverse Fourier transform of $\bar{W}_n(k)$ Eq. (10).

Remark 1. The history of the model and its relatives (e.g., refs. 44 and 48) for the case when $f(\tilde{v}_m)$ is Maxwellian is long. Rayleigh, who wanted to obtain insight into the Boltzmann equation, investigated the limit $\epsilon \ll 1$. This important limit describes dynamics of a heavy Brownian particle in a bath of light gas particles, according to the Rayleigh equation.[40] More recent work considers this model for the case where an external field is acting on the tracer particle, for example in the context of calculation of activation rates over a potential barrier.[42, 43] In the Rayleigh limit of $\epsilon \to 0$ one obtains the dynamics of the Kramers equation, describing Brownian motion in external force field. Investigation of the model for the case where collisions follow a general renewal process (i.e., non-Poissonian) was considered in ref. 45. Related model was recently investigated by Biben *et al.*[46] for inelastic hard spheres M moving in a bath modeled by hard spheres m. The supply of energy from the bath to the inelastic particles mimics dynamics of a driven dissipative system.

2.1. An Example: Lévy Stable Bath Particle Velocities

In the classical works on Brownian motion the condition that bath particle velocity distribution is Maxwellian is imposed. As a result one obtains an equilibrium Maxwellian distribution for the tracer particle (i.e., detailed balance is imposed on the dynamics). This behavior is not unique to Gaussian process, in the sense that if we choose a Lévy stable law to describe the bath particle velocity PDF, the tracer particle will obtain an equilibrium which is also a Lévy distribution. This property does not generally hold for other choices of bath particle velocity PDFs.

To see this let the PDF of bath particle velocities be a symmetric Lévy density, in Fourier space

$$\bar{f}(k) = \exp\left[-\frac{A_\alpha |k|^\alpha}{\Gamma(1+\alpha)}\right], \tag{13}$$

and $0 < \alpha \leqslant 2$. The special case $\alpha = 2$ corresponding to the Gaussian PDF. We will discuss later the dependence of the parameter A_α in Eq. (13) on mass of bath particles m and on a generalized temperature concept. Using Eqs. (8) and (13) we obtain

$$W(V_M, t) = \sum_{n=0}^{\infty} \frac{e^{-Rt}(Rt)^n}{n!} \frac{1}{[Ag_\alpha^n(\epsilon)]^{1/\alpha}} l_\alpha \left\{ \frac{[V_M - V_M(0)\, \xi_1^n]}{[Ag_\alpha^n(\epsilon)]^{1/\alpha}} \right\}, \tag{14}$$

where $l_\alpha(x)$ is the symmetric Lévy density whose Fourier pair is

$$\bar{l}_\alpha(k) = \exp(-|k|^\alpha), \tag{15}$$

$A = A_\alpha / \Gamma(1+\alpha)$, and

$$g_\alpha^n(\epsilon) \equiv \xi_2^\alpha \frac{1-\xi_1^{\alpha n}}{1-\xi_1^\alpha}. \tag{16}$$

Later we will use the $n \to \infty$ limit of Eq. (16)

$$g_\alpha^\infty(\epsilon) = \frac{(2\epsilon)^\alpha}{(1+\epsilon)^\alpha - (1-\epsilon)^\alpha}, \tag{17}$$

and the small ϵ behavior

$$g_\alpha^\infty(\epsilon) \sim \frac{2^{\alpha-1}}{\alpha} \epsilon^{\alpha-1}. \tag{18}$$

From Eq. (14) we see that for all times t and for any mass ratio ϵ, the tracer particle velocity PDF is a sum of rescaled bath particle velocities PDFs. In the limit $t \to \infty$ a stationary state is reached

$$W_{\text{eq}}(V_M) = \frac{1}{[A g_\alpha^\infty(\epsilon)]^{1/\alpha}} l_\alpha \left\{ \frac{V_M}{[A g_\alpha^\infty(\epsilon)]^{1/\alpha}} \right\}, \tag{19}$$

or in Fourier space

$$\bar{W}_{\text{eq}}(k) = \exp[-A g_\alpha^\infty(\epsilon) \, |k|^\alpha]. \tag{20}$$

Thus the distribution of $\tilde{v}_m$ and V_M differ only by a scale parameter. For non-Lévy PDFs of bath particles velocities this is not the case: the distribution of V_M differs from that of $\tilde{v}_m$. Note that for $\alpha = 1$, $g_1^\infty(\epsilon) = 1$ hence the equilibrium velocity distribution Eq. (19) becomes independent of the mass M of the heavy particle (assuming A is independent of M).

3. EQUILIBRIUM

In the long time limit, $t \to \infty$ the tracer particle characteristic function reaches an equilibrium

$$\bar{W}_{\text{eq}}(k) \equiv \lim_{t \to \infty} \bar{W}(k, t). \tag{21}$$

This equilibrium is obtained from Eq. (8). We notice that when $Rt \to \infty$, $P_n(t) = (Rt)^n \exp(-Rt)/n!$ is peaked in the vicinity of $\langle n \rangle = Rt$ hence it is easy to see that

$$\bar{W}_{\text{eq}}(k) = \lim_{n \to \infty} \Pi_{i=1}^{n} \bar{f}(k \xi_1^{n-i} \xi_2). \tag{22}$$

In what follows we investigate properties of this equilibrium.

We will consider the weak collision limit $\epsilon \to 0$. This limit is important since number of collisions needed for the tracer particle to reach an equilibrium is very large. Hence in this case we may expect the emergence of a general equilibrium concept which is not sensitive to the precise details of the velocity PDF $f(\tilde{v}_m)$ of the bath particles.

Remark 1. According to Eq. (10), after a single collision event the PDF of the tracer particle in Fourier space is $\bar{W}_1(k) = \bar{f}(k\xi_2)$ provided that $V_M(0) = 0$. After the second collision event $\bar{W}_2(k) = \bar{f}(k\xi_1\xi_2)\, \bar{f}(k\xi_2)$ and after n collision events

$$\bar{W}_n(k) = \Pi_{i=1}^{n} \bar{f}(k \xi_1^{n-i} \xi_2). \tag{23}$$

This process is described in Fig. 1, where we show $\bar{W}_n(k)$ for $n = 1$, 3, 10, 100, 1000. In this example we use a uniform distribution of the bath particles PDF Eq. (48), with $\epsilon = 0.01$ and $T = 1$. After roughly 100 collision events the characteristic function $\bar{W}_n(k)$ reaches a stationary state, which as we will show is well approximated by a Gaussian (i.e., the Maxwell velocity PDF is obtained).

Remark 2. The equilibrium described by Eq. (22) is valid for a larger class of collision models provided that two requirements are satisfied. To see this consider Eq. (12), this equation is clearly not limited to the model under investigation. For example if number of collisions is described by a renewal process, Eq. (12) is still valid (however generally $P_n(t)$ is not described by the Poisson law). The important first requirement is that in the limit $t \to \infty$ $P_n(t)$ is peaked around $n \to \infty$, and that $P_n(t)$ is not too wide (any renewal process with finite mean time between successive collision events satisfies this condition). The second requirement is that re-collisions of the bath particles and the tracer particle are not important. This assumption is important since without it the simple form of $\bar{W}_n(k)$ is not valid, and hence also Eq. (22). Physically, this means that the bath particles maintain their own equilibrium throughout the collision process, namely we require a fast relaxation to equilibrium of the bath particles.

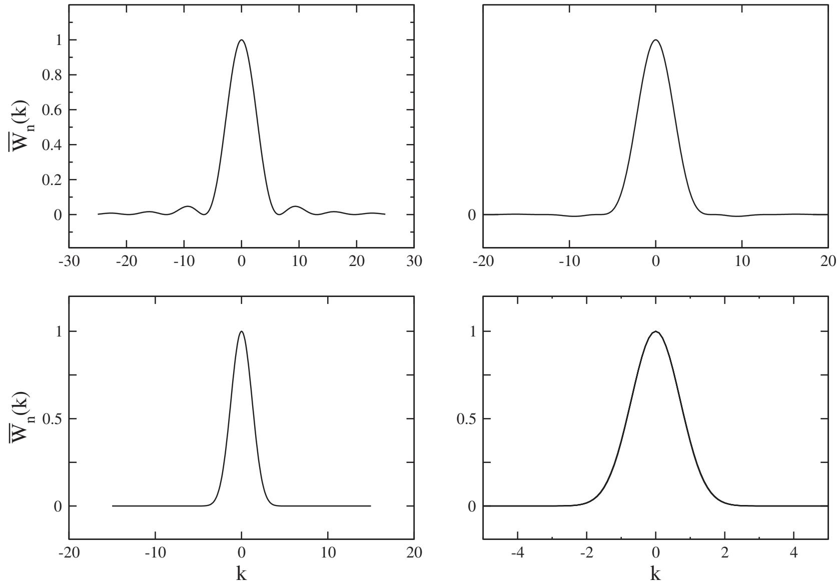

Fig. 1. We show the dynamics of the collision process: the tracer particle characteristic function, conditioned that exactly n collision events have occurred, $\bar{W}_n(k)$ versus k. The velocity PDF of the bath particle is uniform and $\epsilon = 0.01$. We show $n = 1$ (top left), $n = 3$ (top right), $n = 10$ (bottom left) and $n = 100$ $n = 1000$ (bottom right). For the latter case we have $\bar{W}_{100}(k) \simeq \bar{W}_{1000}(k)$, hence the process has roughly converged after 100 collision events. The equilibrium is well approximated with a Gaussian characteristic function indicating that a Maxwell–Boltzmann equilibrium is obtained.

Remark 3. The problem of analysis of the equilibrium Eq. (22) is different from the classical mathematical problem of summation of independent identically distributed random variables.[2] In that case the scaled sum $X_n = \sum_{i=1}^n x_i / n^{1/\alpha}$ in the limit $n \to \infty$ is considered. The characteristic function of X_n is $\bar{\eta}(k/n^{1/\alpha})^n$, where $\bar{\eta}(k)$ is the characteristic function of x_i. In contrast the rescaling of k obtained here in Eq. (22), we have $k \to k\xi_1^{n-i}\xi_2$. This means that (i) we are treating a problem of summation of independent, though nonidentical random variables and (ii) the scaling with n, derived from the dynamics of the model, differs from the $n^{1/\alpha}$ scaling found in the standard problem of summation of random variables.[2]

Remark 4. If $m = M$ we find $\bar{W}_{\mathrm{eq}}(k) = \bar{f}(k)$, this behavior is expected since in this strong collision limit a single collision event is needed for relaxation of tracer particle to equilibrium. This trivial equilibrium is not stable in the sense that perturbing $\bar{f}(k)$ yields a new equilibrium for the tracer particle.

4. MAXWELL VELOCITY DISTRIBUTION

We consider the case where all moments of $f(\tilde{v}_m)$ are finite and that the scaling condition Eq. (1) holds. The second moment of the bath particle velocity is

$$\langle \tilde{v}_m^2 \rangle = \frac{T}{m} \int_{-\infty}^{\infty} x^2 q(x) \, \mathrm{d}x. \tag{24}$$

Without loss of generality we set $\int_{-\infty}^{\infty} x^2 q(x) \, \mathrm{d}x = 1$. The scaling behavior Eq. (1) and the assumption of finiteness of moments of the PDF yields

$$\langle \tilde{v}_m^{2n} \rangle = \left(\frac{T}{m} \right)^n q_{2n}, \tag{25}$$

where the moments of $q(x)$ are defined according to

$$q_{2n} = \int_{-\infty}^{\infty} x^{2n} q(x) \, \mathrm{d}x, \tag{26}$$

and we assume that odd moments of $q(x)$ are zero. Thus the small k expansion of the characteristic function is

$$\bar{f}(k) = 1 - \frac{Tk^2}{2m} + q_4 \left(\frac{T}{m} \right)^2 \frac{k^4}{4!} + O(k^6). \tag{27}$$

For simplicity we consider only the first three terms in the expansion in Eq. (27), we will soon consider the higher order terms in the expansion describing moments beyond the fourth.

We now obtain the velocity distribution of the tracer particle using Eq. (22)

$$\ln[\bar{W}_{\mathrm{eq}}(k)] = \lim_{n \to \infty} \sum_{i=1}^{n} \ln[\bar{f}(k \xi_1^{n-i} \xi_2)]. \tag{28}$$

Inserting Eq. (27) in Eq. (28) we obtain

$$\ln[\bar{W}_{\mathrm{eq}}(k)] = -\frac{T}{2m} g_2^{\infty}(\epsilon) \, k^2 + \frac{q_4 - 3}{4!} \left(\frac{T}{m} \right)^2 g_4^{\infty}(\epsilon) \, k^4 + O(k^6). \tag{29}$$

When ϵ is small we find using Eq. (18)

$$\ln[\bar{W}_{\mathrm{eq}}(k)] = -\frac{Tk^2}{2M} + \left(\frac{T}{M} \right)^2 \frac{q_4 - 3}{4!} 2\epsilon k^4 + O(k^6). \tag{30}$$

It is important to see that the k^4 term approaches zero when $\epsilon \to 0$. Hence we find

$$\lim_{\epsilon \to 0} \ln[\bar{W}_{\text{eq}}(k)] = -\frac{Tk^2}{2M}, \tag{31}$$

inverting to velocity space we obtain the Maxwell velocity PDF

$$\lim_{\epsilon \to 0} W_{\text{eq}}(V_M) = \frac{\sqrt{M}}{\sqrt{2\pi T}} \exp\left(-\frac{MV_M^2}{2T}\right). \tag{32}$$

We see that the parameters q_{2n} with $n > 1$ are the irrelevant parameters of the problem, and hence the Maxwell distribution is stable in the sense that it does not depend on the detailed shape of $f(\tilde{v}_m)$.

To complete the proof we will show that the k^6, k^8 and higher order terms in Eq. (30) also approach zero when $\epsilon \to 0$. Let $\tilde{\kappa}_{m,2n}$ ($\kappa_{M,2n}$) be the $2n$th cumulant of bath particle (tracer particle) velocity. The cumulants describing the bath particle are related to the moments q_{2n} in the usual way $\tilde{\kappa}_{m,2} = T/m$, $\tilde{\kappa}_{m,4} = (q_4 - 1)(T/m)^2$, etc. Then using Eq. (22) one can show that

$$\kappa_{M,2n} = g_{2n}^{\infty}(\epsilon)\, \tilde{\kappa}_{m,2n}. \tag{33}$$

From the scaling function Eq. (1) we have $\tilde{\kappa}_{m,2n} = c_{2n} T^n / m^n$, where c_{2n} are dimensionless parameters which depend on $f(\tilde{v}_m)$, $n = 1, 2,\ldots$, e.g., $c_2 = 1$, $c_4 = q_4 - 1$, etc. The parameters c_{2n} for $n > 1$ are the irrelevant parameters of the model in the limit of weak collisions. To see this note that when $\epsilon \to 0$ we have

$$\kappa_{M,2n} = (T_2/M)\, \delta_{n1}. \tag{34}$$

Thus, besides the second cumulant, all cumulants of the tracer particle velocity distribution function are zero. As well known the cumulants of the Gaussian PDF with zero mean are all zero besides second. Equation (34) shows that the tracer particle reached the Maxwell equilibrium.

Remark 1. Consider the case where the second moment of bath particles velocity $\langle \tilde{v}_m^2 \rangle$ is finite, but that higher order moments diverge. Using the scaling condition Eq. (1) the small k expansion of the bath particle characteristic function is

$$\bar{f}(k) \sim 1 - \frac{Tk^2}{2m} + q_\beta \left(\frac{T}{m}\right)^{\beta/2} \frac{|k|^\beta}{\Gamma(1+\beta)} + O(k^4) \tag{35}$$

and $2 < \beta < 4$. Using Eqs. (18) and (28), we find in the limit of small ϵ

$$\ln[\bar{W}_{\rm eq}(k)] = -\frac{Tk^2}{2M} + \frac{q_\beta}{\Gamma(1+\beta)}\left(\frac{T}{M}\right)^{\beta/2}\frac{2^{\beta-1}}{\beta}\,\epsilon^{\beta/2-1}\,|k|^\beta + O(k^4). \tag{36}$$

The interesting thing to notice is that the $|k|^\beta$ term approaches zero when $\epsilon \to 0$. Hence also for this case Maxwell velocity distribution is obtained for the tracer particle. Note that in this case moments of the velocity of the tracer particle are not described well by moments of the Maxwell distribution (i.e., deviation between Maxwell distribution, and exact velocity distribution, are expected in the tails of the velocity distribution $|V_M| \to \infty$). Note that similar behavior is found for the Gaussian Central Limit Theorem,[2] convergence of the sum of independent, identically, distributed random variables, to Gaussian law is obtained if the second moment of the random variables is finite (i.e., higher order moments may diverge).

Remark 2. As expected when the scaling condition Eq. (1) is not satisfied we do not obtain the Maxwell equilibrium. We may ask if Gaussian (though not necessarily Maxwellian) velocity PDFs are obtained if we do not impose the scaling condition. We note that it is not sufficient to demand that velocity PDF $f(\tilde{v}_m)$ has finite second moment (or even all moments) to obtain a Gaussian equilibrium. An example is the case $\langle \tilde{v}_m^4 \rangle \propto 1/m^3$ where the k^4 term survives the limit $\epsilon \to 0$.

5. THERMODYNAMIC APPROACH

In this section the thermodynamic approach is used, imposing the condition that the equilibrium of the tracer particle, i.e., $\bar{W}_{\rm eq}(k)$, is independent of the mass of the bath particles m. As well known the stationary velocity distribution of a Brownian particle in thermal equilibrium with its surrounding fluid is independent of the mass of the bath particles. Our aim is to find the conditions for such a behavior for a wider class of equilibrium (i.e., beyond the Maxwell distribution). We call this approach a thermodynamic approach since a thermodynamic system A (the tracer particle) left in thermal contact with system B (the bath particles) obtains a thermal equilibrium which does not depend on the coupling constant to the bath or the mass of the bath particles.

5.1. Lévy Equilibrium

We now assume that the bath particle velocity PDF is even with zero mean, and that it decays like a power law $P(\tilde{v}_m) \propto |\tilde{v}_m|^{-(1+\alpha)}$ when $\tilde{v}_m \to \infty$

where $0 < \alpha < 2$. In this case the variance of the bath particle velocity distribution diverges. We use the small k expansion of the bath particle characteristic function

$$\bar{f}(k) = 1 - \frac{A_\alpha |k|^\alpha}{\Gamma(1+\alpha)} + \frac{B |k|^\beta}{\Gamma(1+\beta)} + o(|k|^\beta) \tag{37}$$

with $\alpha < \beta \leqslant 2\alpha$. Using Eq. (28) we find

$$\ln[\bar{W}_{\text{eq}}(k)] = \begin{cases} -\frac{A_\alpha}{\Gamma(1+\alpha)} g_\alpha^\infty(\epsilon) |k|^\alpha + \frac{B}{\Gamma(1+\beta)} g_\beta^\infty(\epsilon) |k|^\beta & \beta < 2\alpha \\ -\frac{A_\alpha}{\Gamma(1+\alpha)} g_\alpha^\infty(\epsilon) |k|^\alpha + \left[\frac{B}{\Gamma(1+2\alpha)} - \frac{A^2}{2\Gamma^2(1+\alpha)} \right] g_{2\alpha}^\infty(\epsilon) |k|^{2\alpha} & \beta = 2\alpha \end{cases} \tag{38}$$

where terms of order higher than $|k|^\beta$ are neglected. Using Eq. (18) we obtain in the limit $\epsilon \to 0$

$$\ln[\bar{W}_{\text{eq}}(k)] \sim \begin{cases} -\frac{A_\alpha}{\Gamma(1+\alpha)} \left(\frac{m}{M}\right)^{\alpha-1} \frac{2^{\alpha-1}}{\alpha} |k|^\alpha \\ \quad + \frac{B}{\Gamma(1+\beta)} \left(\frac{m}{M}\right)^{\beta-1} \frac{2^{\beta-1}}{\beta} |k|^\beta & \beta < 2\alpha \\ -\frac{A_\alpha}{\Gamma(1+\alpha)} \left(\frac{m}{M}\right)^{\alpha-1} \frac{2^{\alpha-1}}{\alpha} |k|^\alpha \\ \quad + \left[\frac{B}{\Gamma(1+2\alpha)} - \frac{A^2}{2\Gamma^2(1+\alpha)} \right] \left(\frac{m}{M}\right)^{2\alpha-1} \frac{2^{2\alpha-1}}{2\alpha} |k|^{2\alpha} & \beta = 2\alpha. \end{cases} \tag{39}$$

To obtain an equilibrium we use the thermodynamical argument. We require that equilibrium velocity PDF of the tracer particle M be independent of the mass of the bath particle m. This condition yields

$$A_\alpha = \frac{T_\alpha}{m^{\alpha-1}}, \tag{40}$$

where we are forced to introduce the generalized T_α, whose units are $\text{Kg}^{\alpha-1}\text{Mt}^\alpha/\text{Sec}^\alpha$. An additional requirement is needed to obtain a unique equilibrium (i.e., an equilibrium which does not depend on details of bath particle velocity PDF like B): terms beyond the $|k|^\alpha$ term in Eq. (39) must vanish in the limit $\epsilon \to 0$. This occurs for bath particles velocity PDFs $f(\tilde{v}_m)$ which satisfy the condition

$$B \propto \frac{1}{m^{\theta}} \tag{41}$$

and $\theta < \beta - 1$. Using this condition we obtain

$$\lim_{\epsilon \to 0} \ln[\bar{W}_{\mathrm{eq}}(k)] = -\frac{T_{\alpha} 2^{\alpha-1}}{\Gamma(1+\alpha)\, \alpha M^{\alpha-1}} |k|^{\alpha}, \tag{42}$$

hence Lévy type of equilibrium is obtained

$$\lim_{\epsilon \to 0} W_{\mathrm{eq}}(V_M) = \left[\frac{\Gamma(1+\alpha)\, M^{*(\alpha-1)}}{T_{\alpha}}\right]^{1/\alpha} l_{\alpha} \left\{\left[\frac{\Gamma(1+\alpha)\, M^{*(\alpha-1)}}{T_{\alpha}}\right]^{1/\alpha} V_M\right\} \tag{43}$$

where $M^* = M\alpha^{1/(\alpha-1)}/2$ is the renormalized mass. For $\alpha = 2$ the Maxwell PDF Eq. (32) is recovered and $M^* = M$.

From physical requirements we note that our results are valid only when $1 < \alpha \leqslant 2$. If $\alpha = 1$ the velocity PDF becomes independent of the mass of the tracer particle, while when $\alpha < 1$ the heavier the tracer particle the faster its motion (in statistical sense). This implies that imposing the condition of independence of the heavy tracer particle velocity PDF on the mass of the bath particles m, based on the thermodynamic argument, is most likely not the correct path.

Remark 1. The domain of attraction of the Lévy equilibrium we find, Eq. (43) does not include all power law distribution with $\alpha < 2$. Consider for example

$$\bar{f}(k) = \frac{1}{2}\left\{\exp\left[-\frac{2T_{\alpha}\, |k|^{\alpha}}{m^{\alpha-1}\Gamma(1+\alpha)}\right] + \exp\left(-\frac{T_2 k^2}{m}\right)\right\}, \tag{44}$$

hence the gas particle velocity distribution in this case is a sum of a Gaussian and a Lévy distributions. The small k expansion of Eq. (44) is

$$\bar{f}(k) = 1 - \frac{T_{\alpha}\, |k|^{\alpha}}{m^{\alpha-1}\Gamma(1+\alpha)} - \frac{T_2 k^2}{2m} \cdots, \tag{45}$$

where we consider $1 < \alpha < 2$. Using Eqs. (37) and (45) we find $\beta = 2$, while comparing Eq. (41) and Eq. (45) yields $\theta = 1$. Now the condition $\theta < \beta - 1$ does not hold, and hence the Lévy equilibrium Eq. (43) is not obtained.

Interestingly, one can show that for this case one obtains an equilibrium characteristic function $\bar{W}_{\text{eq}}(k)$ which is a convolution of a Lévy PDF and a Gaussian PDF. I suspect that this type of equilibrium is not limited to this example.

5.2. Numerical Examples

We now investigate numerically exact solutions of the problem, and compare these solutions to the stable equilibrium which becomes exact when $\epsilon \to 0$. Our numerical examples yield: (i) information on the convergence rate to stable equilibrium, and (ii) they also investigate the question what finite small values of ϵ yield an equilibrium which is well approximated by a stable equilibrium.

5.2.1. Maxwell Statistics

First consider the Maxwellian case. We investigate three types of bath particle velocity PDFs:

(i) The exponential

$$f(\tilde{v}_m) = \frac{\sqrt{2m}}{2\sqrt{T_2}} \exp\left(-\frac{\sqrt{2m}\,|\tilde{v}_m|}{\sqrt{T_2}}\right), \tag{46}$$

which yields

$$\bar{f}(k) = \frac{1}{1 + \frac{T_2 k^2}{2m}}. \tag{47}$$

(ii) The uniform PDF

$$f(\tilde{v}_m) = \begin{cases} \sqrt{\frac{m}{12T_2}} & \text{if} \quad |\tilde{v}_m| < \sqrt{\frac{3T_2}{m}} \\ 0 & \text{otherwise} \end{cases} \tag{48}$$

which yields

$$\bar{f}(k) = \frac{\sin\left(\sqrt{\frac{3T_2}{m}}\,k\right)}{\sqrt{\frac{3T_2}{m}}\,k}. \tag{49}$$

(iii) The Gaussian PDF

$$\bar{f}(k) = \exp\left(-\frac{k^2 T_2}{2m}\right). \tag{50}$$

The small k expansion of Eqs. (47,49,50) is $\bar{f}(k) \sim 1 - k^2 T_2/(2m) + \cdots$, indicating that the second moment of velocity of bath particles $\langle \tilde{v}_m^2 \rangle$ is identical for the three PDFs.

To obtain numerically exact solution of the problem we use Eq. (22) with large though finite n. In all our numerical examples we used $M = 1$ hence $m = \epsilon$. Thus for example for the uniform velocity PDF Eq. (49) we have

$$\bar{W}_{\text{eq}}(k) \simeq \exp\left\{\sum_{i=1}^{n} \ln\left[\sqrt{\frac{\epsilon}{3T_2}} \frac{\sin\left(k\sqrt{\frac{3T_2}{m}}\left(\frac{1-\epsilon}{1+\epsilon}\right)^{n-i}\frac{2\epsilon}{1+\epsilon}\right)}{k\left(\frac{1-\epsilon}{1+\epsilon}\right)^{n-i}\frac{2\epsilon}{1+\epsilon}}\right]\right\}. \tag{51}$$

To obtain equilibrium we increase n for a fixed ϵ and temperature until a stationary solution is obtained.

According to our analytical results the bath particle velocity PDFs Eqs. (46), (48), and (50), belong to the domain of attraction of the Maxwellian equilibrium. In Fig. 2 we show $\bar{W}_{\text{eq}}(k)$ obtained from numerical solution of the problem. The numerical solution exhibits an excellent agreement with Maxwell's equilibrium. Thus details of the precise shape of velocity PDF of bath particles are unimportant, and as expected the Maxwell distribution is stable. We note that the convergence rate to equilibrium depends on the value of k. To obtain the results in Fig. 2, I used $\epsilon = 0.01$, $T_2 = 2$, $M = 1$, and $n = 2000$. For examples shown below, much larger values of n and smaller values of ϵ, are needed to obtain a good fit to the Lévy equilibrium.

5.2.2. Lévy Statistics

We now consider four power law PDFs satisfying $f(\tilde{v}_m) \propto |\tilde{v}_m|^{-5/2}$, namely $\alpha = 3/2$.

(i) Case 1 we choose

$$f(\tilde{v}_m) = \frac{N_1}{(1 + c_1 |\tilde{v}_m|)^{5/2}}, \tag{52}$$

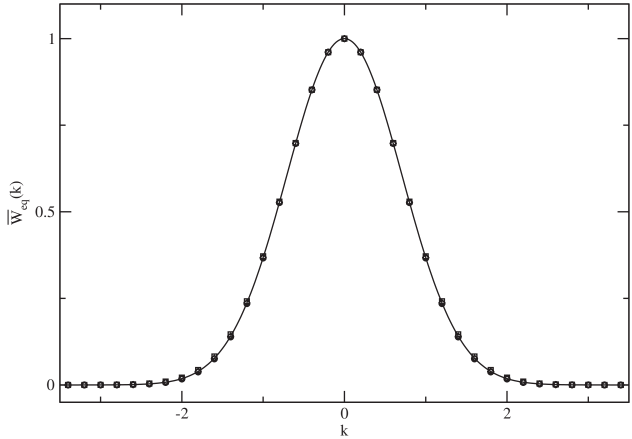

Fig. 2. The equilibrium characteristic function of the tracer particle, $\overline{W}_{eq}(k)$ versus k. We consider three types of bath particles velocity PDFs (i) exponential (squares), (ii) uniform (circles), and (iii) Gaussian (diamonds). The velocity distribution of the tracer particle M is well approximated by Maxwell's distribution plotted as the solid curve $\overline{W}_{eq}(k) = \exp(-|k|^2)$. For the numerical results I used: $M = 1$, $T_2 = 2$, $n = 2000$, and $\epsilon = 0.01$.

with $c_1 = (3\pi)^{2/3} m^{1/3} 2^{-1} (T_{3/2})^{2/3}$ and $N_1 = 3c_1/4$. The characteristic function is given in terms of a generalized Hyper-geometric function[47]

$$\bar{f}(k) = \frac{4N_1}{3c_1^{5/2}} \left\{ c_1^{3/2} \, {}_1F_2\left(1; -\frac{1}{4}, \frac{1}{4}; -\frac{k^2}{4c_1^2}\right) - |k|^{3/2} \sqrt{2\pi} \left[\cos\left(\frac{k}{c_1}\right) + \mathrm{Sign}(k) \sin\left(\frac{k}{c_1}\right) \right] \right\}. \tag{53}$$

(ii) Case 2

$$f(\tilde{v}_m) = \frac{N_2}{1 + c_2 |\tilde{v}_m|^{5/2}}, \tag{54}$$

where $c_2 = (2\sqrt{2\pi m})^{5/3} (3GT_{3/2})^{-5/3}$, $G = (4\pi/5)\sqrt{2/(5+\sqrt{5})}$, $N_2 = c_2^{2/5}/(2G)$. The characteristic function can be formally expressed in terms of a Meijer G function[47] (not shown). For our numerical results we used the numerical Fourier transform of Eq. (54) to obtain $\overline{W}_{eq}(k)$.

(iii) Case 3

$$f(\tilde{v}_m) = \frac{N_3}{(1 + C_3 \tilde{v}_m^2)^{5/4}} \tag{55}$$

where $C_3 = \{\Gamma[1/4]\,\Gamma[5/2]\sqrt{2m}\,3^{-1}T_{3/2}^{-1}\Gamma^{-1}(3/4)]\}^{4/3}$, $N_3 = 0.75T_{3/2} \times \Gamma^{-1}(5/2)(2\pi m)^{-1/2}\,C_3^{5/4}$. The characteristic function is

$$\bar{f}(k) = N_3 C_3^{-7/8}\,\frac{2^{1/4}\sqrt{\pi}}{\Gamma(\frac{5}{4})}\,|k|^{3/4}\,K_{3/4}\left(\frac{|k|}{\sqrt{C_3}}\right), \tag{56}$$

where $K_{3/4}$ is the modified Bessel function of the second kind.

(iv) Case 4 the Lévy PDF with index 3/2 whose Fourier pair is

$$\bar{f}(k) = \exp\left[-\frac{T_{3/2}\,|k|^{5/2}}{\sqrt{m}\,\Gamma(5/2)}\right]. \tag{57}$$

The corresponding PDF is also called Holtsmark PDF.

The $|\tilde{v}_m| \to \infty$ behavior of the PDFs (52)–(57) is

$$f(\tilde{v}_m) \sim \frac{3T_{3/2}}{4\sqrt{2\pi m}}\,|\tilde{v}_m|^{-5/2}, \tag{58}$$

hence the small k behavior of the characteristic function is

$$\bar{f}(k) \sim 1 - \frac{T_{3/2}\,|k|^{3/2}}{\sqrt{m}\,\Gamma(5/2)}. \tag{59}$$

According to our results in previous section the velocity PDFs Eqs. (52)–(54) belong to the domain of attraction of the Lévy type of equilibrium

$$\lim_{\epsilon \to 0} \bar{W}_{\text{eq}}(k) = \exp\left[-\frac{T_{3/2}2^{3/2}}{\sqrt{M}\,3\Gamma(5/2)}\,|k|^{3/2}\right]. \tag{60}$$

Namely, $W_{\text{eq}}(V_M)$ is the Lévy PDF with index 3/2 also called the Holtsmark PDF.

Similar to the Maxwellian case, numerically exact solution are obtained, in k space using Eq. (22) with finite n. For example for the power law velocity PDF Eq. (55) we use

$$\bar{W}_{\text{eq}}(k) \simeq \exp\left\{\sum_{i=1}^{n} \log\left[\frac{N_3}{C_3^{7/8}}\,\frac{\sqrt{\sqrt{2}\,\pi}}{\Gamma(5/4)}\,|k|^{3/4}\left(\frac{1-\epsilon}{1+\epsilon}\right)^{3(n-i)/4}\left(\frac{2\epsilon}{1+\epsilon}\right)^{3/4}\right.\right.$$
$$\left.\left.\times K_{3/4}\left(\left(\frac{1-\epsilon}{1+\epsilon}\right)^{n-i}|k|\,\frac{2\epsilon}{1+\epsilon}\,(C_3)^{-1/2}\right)\right]\right\}, \tag{61}$$

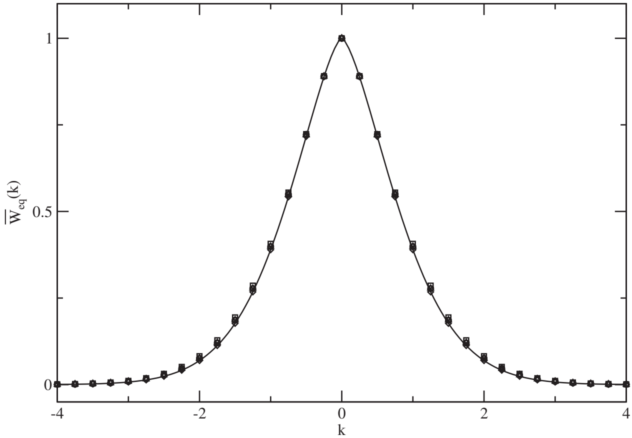

Fig. 3. The equilibrium characteristic function $\bar{W}_{\rm eq}(k)$, for the case when velocities of bath particles are distributed according to a power law with $\alpha = 3/2$. Four types of bath particles characteristic functions are considered: (i) the generalized Hyper-geometric function Eq. (52) (squares), (ii) the Meijer G function Eq. (54), (circles), (iii) Bessel function Eq. (55), (triangles), and (iv) the Holtsmark function Eq. (57) (diamonds). For all these cases, the equilibrium characteristic function $\bar{W}_{\rm eq}(k)$ is well approximated by the Lévy characteristic function; the solid curve $\bar{W}_{\rm eq}(k) = \exp(-2^{3/2}\,|k|^{3/2}/3)$. For the numerical results we used: $M = 1$, $T_\alpha = \Gamma(5/2)$, $\epsilon = 5e-5$, and $n = 1e6$.

where we fix ϵ and $T_{3/2}$ and increase n until a stationary solution is obtained.

In Fig. 3 we show $\bar{W}_{\rm eq}(k)$ obtained using numerically exact solution based on the four PDFs Eqs. (52), (54), (55), and (57). The exact solutions are in good agreement with the theoretical prediction Eq. (60). Thus, similar to the Maxwellian case, the exact shape of the equilibrium distribution does not depend on the details of the velocity PDF of the bath particles (besides α and T_α of course).

We note that the convergence towards the stable equilibrium was found to be slow if compared with the Gaussian case. For example for the Bessel function Eq. 56 and for $\epsilon = 1e-6$, I obtained (what I judge as reasonable) convergence only when $n > 3e6$ (for $-4 < k < 4$). The convergence rate to equilibrium depends on k, and as expected is faster for small values of k.

We also consider the marginal case $\alpha = 1$, which marks the transition from finite $\langle|\tilde{v}_m|\rangle$ for $\alpha > 1$ to infinite value of $\langle|\tilde{v}_m|\rangle$ for $\alpha < 1$. We considered the velocity PDF

$$f(\tilde{v}_m) = \frac{1}{2(1+|\tilde{v}_m|)^2}. \tag{62}$$

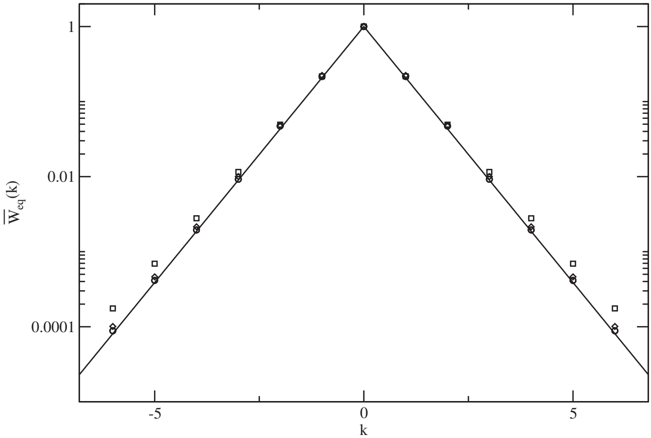

Fig. 4. The equilibrium characteristic function, for a case when the velocity of the bath particles is a power law with $\alpha = 1$ Eq. (62). We use $\epsilon = 0.01$ (squares), $\epsilon = 0.001$ (diamonds), $\epsilon = 0.0005$ (circles). As ϵ approaches zero, number of collisions needed to reach an equilibrium becomes very large and then a Lévy type of thermal equilibrium is obtained: the solid curve $\bar{W}_{eq}(k) = \exp(-\pi\,|k|/2)$. Note the logarithmic scale of the figure.

The characteristic function is expressed in terms of a Meijer G function using Mathematica[47]

$$\bar{f}(k) = \frac{1}{\sqrt{\pi}} G^{31}_{00}\left(\frac{k^2}{4} \middle| \begin{matrix} 0 \\ 0, \frac{1}{2}, 1 \end{matrix} \right). \tag{63}$$

The small k behaves of the characteristic function is $\bar{f}(k) = 1 - \pi\,|k|/2 \cdots$. According to the theory in the limit $\epsilon \to 0$

$$\bar{W}_{eq}(k) = \exp(-\pi\,|k|/2). \tag{64}$$

Thus the velocity PDF of the tracer particle M is Lorentzian. As mentioned in this case the equilibrium obtained is independent of mass M. In Fig. 4 we show the numerically exact solution of $\bar{W}_{eq}(k)$ for several mass ratios ϵ. As $\epsilon \to 0$ we obtain the predicted Lévy type of equilibrium.

6. SCALING APPROACH: THE RELEVANT SCALE IS ENERGY

We now return to the energy scaling approach. We assume that the statistical properties of the bath particles velocities can be characterized with an energy scale T. Since T, m, and $\tilde{v}_m$ are the only variables describing the bath particles. The PDF of velocities of bath particles is given by Eq. (1). We also assume that $f(\tilde{v}_m)$ is an even function, as expected from symmetry. The scaling assumption made in Eq. (1) is very natural, since the total energy of bath particles is nearly conserved, i.e., the energy transfer to the single heavy particle being much smaller than the total energy of the bath particles.

6.1. Lévy Velocity Distribution

Now we assume a power law behavior of $f(\tilde{v}_m)$ i.e., $q(x) \propto |x|^{-(1+\alpha)}$ when $|x| \to \infty$ and $0 < \alpha < 2$, where $q(x)$ is the scaling function defined in Eq. (1). This scaling function satisfies the normalization condition

$$\int_{-\infty}^{\infty} q(x)\,\mathrm{d}x = 1. \tag{65}$$

For this power law case the variance of the bath particles velocity is infinite. For this case the bath particle characteristic function is

$$\bar{f}(k) = 1 - \frac{q_\alpha}{\Gamma(1+\alpha)} \left(\frac{T}{m}\right)^{\alpha/2} |k|^\alpha + \frac{q_\beta}{\Gamma(1+\beta)} \left(\frac{T}{m}\right)^{\beta/2} |k|^\beta + o(|k|^\beta) \tag{66}$$

where $\alpha < \beta \leqslant 2\alpha$. q_α and q_β are dimensionless numbers which depend of course on $q(x)$. Without loss of generality we may set $q_\alpha = 1$. In Eq. (66) we have used the assumption that $f(\tilde{v}_m)$ is even.

Using the same technique used in previous section we obtain the equilibrium characteristic function for the tracer particle M

$$\ln[\bar{W}_{\mathrm{eq}}(k)] = \begin{cases} -\dfrac{1}{\Gamma(1+\alpha)}\left(\dfrac{T}{m}\right)^{\alpha/2} g_\alpha^\infty(\epsilon)\,|k|^\alpha \\ \quad + \dfrac{q_\beta}{\Gamma(1+\beta)}\left(\dfrac{T}{m}\right)^{\beta/2} g_\beta^\infty(\epsilon)\,|k|^\beta & \beta < 2\alpha \\ -\dfrac{1}{\Gamma(1+\alpha)}\left(\dfrac{T}{m}\right)^{\alpha/2} g_\alpha^\infty(\epsilon)\,|k|^\alpha \\ \quad + \left[\dfrac{q_{2\alpha}}{\Gamma(1+2\alpha)} - \dfrac{1}{2\Gamma^2(1+\alpha)}\right]\left(\dfrac{T}{m}\right)^{\alpha} g_{2\alpha}^\infty(\epsilon)\,|k|^{2\alpha} & \beta = 2\alpha. \end{cases} \tag{67}$$

Taking the small ϵ limit the following expansions are found, if $\beta < 2\alpha$

$$\ln[\bar{W}_{\text{eq}}(k)] \sim -\frac{2^{\alpha-1}}{\alpha\Gamma(1+\alpha)}\left(\frac{T}{M}\right)^{\alpha/2}\frac{|k|^{\alpha}}{\epsilon^{1-\alpha/2}} + \frac{q_{\beta}2^{\beta-1}}{2\Gamma(1+\beta)}\left(\frac{T}{M}\right)^{\beta/2}\frac{|k|^{\beta}}{\epsilon^{1-\beta/2}} + o(|k|)^{\beta}, \tag{68}$$

if $\beta = 2\alpha$

$$\ln[\bar{W}_{\text{eq}}(k)] \sim -\frac{2^{\alpha-1}}{\alpha\Gamma(1+\alpha)}\left(\frac{T}{M}\right)^{\alpha/2}\frac{|k|^{\alpha}}{\epsilon^{1-\alpha/2}} + \left[\frac{q_{\beta}}{\Gamma(1+\beta)} - \frac{1}{2\Gamma^{2}(1+\alpha)}\right]\left(\frac{T}{M}\right)^{\alpha}\frac{2^{2\alpha-1}}{2\alpha}\frac{|k|^{2\alpha}}{\epsilon^{1-\alpha}} + o(|k|^{2\alpha}). \tag{69}$$

Thus for example if $\beta = 2$ the leading term in Eq. (68) scales with ϵ according to $\epsilon^{\alpha/2-1} \to \infty$, while the second term scales like ϵ^{0}. Thus from Eqs. (68) and (69) we see when ϵ is small and k not too large, we may neglect the second and similarly higher order terms. This yields the equilibrium of the test particle which is a stretched exponential in k space

$$\bar{W}_{\text{eq}}(k) \sim \exp\left[-\frac{2^{\alpha-1}}{\alpha\Gamma(1+\alpha)}\left(\frac{T}{M}\right)^{\alpha/2}\frac{|k|^{\alpha}}{\epsilon^{1-\alpha/2}}\right], \tag{70}$$

and hence the equilibrium velocity distribution of the tracer particle is Lévy stable. We note that for $\alpha \neq 2$ the equilibrium Eq. (70) depends on ϵ, and hence on the mass of the bath particles m. While for the Maxwell case $\alpha = 2$, the equilibrium is independent of the coupling constant ϵ. This difference between the Lévy equilibrium and the Maxwell equilibrium is related to the conservation of energy and momentum during a collision event, and to the fact that the energy of the particles is quadratic in their velocities. The asymptotic behavior Eq. (70) is now demonstrated using numerical examples.

6.2. Numerical Examples: Lévy Equilibrium

We consider three types of bath particle velocity PDFs, for large values of $|v_m| \to \infty$ these PDFs exhibit $f(\tilde{v}_m) \propto |\tilde{v}_m|^{-5/2}$, which implies $\alpha = 3/2$.

(i) Case 1

$$f(\tilde{v}_m) = \frac{N_1}{\left(1 + c\sqrt{\frac{m}{T}}\,|\tilde{v}_m|\right)^{5/2}}, \tag{71}$$

where the normalization constant is $N_1 = c3\sqrt{m/T}/4$ and $c = 3^{2/3}/2$. The Fourier transform of this equation can be expressed in terms of a Hypergeometric function as in Eq. (53).

(ii) Case 2,

$$f(\tilde{v}_m) = \frac{N_2}{\left(1 + \frac{m\tilde{v}_m^2}{2\tilde{T}}\right)^{5/4}} \tag{72}$$

for $-\infty < \tilde{v}_m < \infty$. The normalization constant is

$$N_2 = \sqrt{\frac{m}{2\tilde{T}}}\,\frac{\Gamma(\frac{1}{4})}{4\sqrt{\pi}\,\Gamma(\frac{3}{4})}, \tag{73}$$

and

$$\tilde{T} = \frac{2}{\pi^{2/3}}\left[\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\right]^{4/3} T. \tag{74}$$

The bath particle characteristic function is

$$\bar{f}(k) = \frac{2^{1/4} N\sqrt{\pi}}{\Gamma(5/4)\left(\frac{m}{2\tilde{T}}\right)^{7/8}}\,|k|^{3/4}\,K_{3/4}\left(\frac{k\sqrt{2\tilde{T}}}{\sqrt{m}}\right), \tag{75}$$

which yields the small k expansion

$$\bar{f}(k) = 1 - 2.34565\left(\frac{\tilde{T}}{m}\right)^{3/4}|k|^{3/2} + 2\,\frac{\tilde{T}}{m}\,k^2 + \cdots, \tag{76}$$

or using Eq. (74) $\bar{f}(k) = 1 - (T/m)^{3/4}\,|k|^{3/2}/\Gamma(5/2) + \cdots$.

(iii) Case 3, the bath particle velocity PDF is a Lévy PDF with index $3/2$, whose characteristic function is

$$\bar{f}(k) = \exp\left[-\left(\frac{T}{m}\right)^{3/4}\frac{|k|^{3/2}}{\Gamma(5/2)}\right]. \tag{77}$$

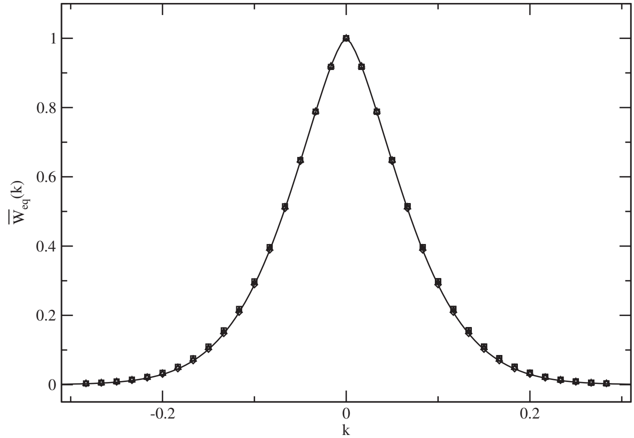

Fig. 5. We show the equilibrium characteristic function of the tracer particle, using the energy scaling approach. Numerically exact solution of the problem are obtained using three long tailed bath particle velocity PDFs (i) Eq. (71) squares, (ii) Eq. (72) triangles, (iii) Eq. (77) diamonds. The tracer particle equilibrium is well approximated by the Lévy characteristic function the solid curve; $\overline{W}_{\text{eq}}(k) \sim \exp(-2.211\,|\frac{k}{\epsilon^{1/6}}|^{3/2})$. For the numerical results we used $T = 4.555$, $\epsilon = 1e-5$, and $M = 1$.

According to our theory these power law velocity PDFs, yield a Lévy equilibrium for the tracer particle, Eq. (70). In Fig. 5 we show numerically exact solution of the problem for cases (1)–(3). These solutions show a good agreement between numerical results and the asymptotic theory. The Lévy equilibrium for the tracer particle is not sensitive to precise shape of the velocity distribution of the bath particle, and hence like the Maxwellian equilibrium is stable.

7. SUMMARY AND DISCUSSION

The main points of this manuscript and some possible extensions are:

(i) Our work shows that if a heavy tracer particle is inserted in a steady state bath, the tracer particle reaches a stable equilibrium Eqs. (43) and (70). The exponent describing the power law tail of the bath particle velocity distribution yields the Lévy exponent α.

(ii) The domain of attraction of the stable Lévy type of equilibrium was clarified. The problem is different from the classical mathematical problem of summation of independent identically distributed random variables. For example, a case where the equilibrium is a convolution of the Lévy and the Gaussian distributions was briefly pointed out.

(iii) Kinetic models can be used to obtain stable power law generalizations of the Maxwell–Boltzmann equilibrium, without assuming inelastic collisions.

(iv) I recently showed[32] that Lévy equilibrium is obtained also from a nonlinear Boltzmann equation approach, for a driven Maxwell gas in the elastic limit.

(v) It is worth while checking if Lévy equilibrium can be obtained also from other kinetic models. Specifically our model assumes (a) a uniform collision rate, (b) dynamics of the tracer particle is driven by energy supply from bath particles, and (c) the collisions are elastic. It would be interesting to check what happens when these assumptions are relaxed.

(vi) The thermodynamic approach yields a physical result only for $1 < \alpha$. Within this approach the equilibrium of the tracer particle is independent of the mass of the bath particles. The approach is based on a generalized temperature concept.

(vii) The energy scaling approach is based on standard temperature concept T. However the temperature does not yield the averaged energy of the heavy test particle (which is infinite), instead it yields a measure of the width of the velocity distribution. Within this approach the Lévy equilibrium for the tracer particle depends on the mass of the bath particles. Thus unlike ordinary equilibrium statistical mechanics the velocity distribution of the heavy particle depends on the details of the interaction with the bath.

APPENDIX A

In this Appendix the solution of the equation of motion for $\bar{W}(k,t)$ Eq. (7) is obtained, the initial condition is $\bar{W}(k,0) = \exp[ikV_M(0)]$. The inverse Fourier transform of this solution yields $W(V_M, t)$ with initial condition $W(V_M, 0) = \delta[V_M - V_M(0)]$. Such a solution is obtained in Eq. (14), for the special case when $f(\tilde{v}_m)$ is a Lévy PDF.

Introduce the Laplace transform

$$\bar{W}(k,s) = \int_0^\infty \bar{W}(k,t)\exp(-st)\,\mathrm{d}t. \tag{78}$$

Using Eq. (7) we have

$$s\bar{W}(k,s) - e^{ikV_M(0)} = -R\bar{W}(k,s) + R\bar{W}(k\xi_1, s)\,\bar{f}(k\xi_2), \tag{79}$$

this equation can be rearranged to give

$$\bar{W}(k,s)=\frac{e^{ikV_M(0)}}{R+s}+\frac{R}{R+s}\bar{W}(k\xi_1,s)\,\bar{f}(k\xi_2). \tag{80}$$

This equation is solved using the following procedure. Replace k with $k\xi_1$ in Eq. (80)

$$\bar{W}(k\xi_1,s)=\frac{e^{ik\xi_1V_M(0)}}{R+s}+\frac{R}{R+s}\bar{W}(k\xi_1^2,s)\,\bar{f}(k\xi_2\xi_1). \tag{81}$$

Equation (81) may be used to eliminate $\bar{W}(k\xi_1, s)$ from Eq. (80), yielding

$$\bar{W}(k,s)=\frac{e^{ikV_M(0)}}{R+s}+\frac{Re^{ik\xi_1V_M(0)}}{(R+s)^2}\,\bar{f}(k\xi_2)+\frac{R^2}{(R+s)^2}\bar{W}(k^2\xi_1,s)\,\bar{f}(k\xi_2\xi_1)\,\bar{f}(k\xi_2). \tag{82}$$

Replacing k with $k\xi_1^2$ in Eq. (80)

$$\bar{W}(k\xi_1^2,s)=\frac{e^{ik\xi_1^2V_M(0)}}{R+s}+\frac{R}{R+s}\bar{W}(k\xi_1^3,s)\,\bar{f}(k\xi_2\xi_1^2). \tag{83}$$

Inserting Eq. (83) in Eq. (82) and rearranging

$$\bar{W}(k,s)=\frac{e^{ikV_M(0)}}{R+s}+\frac{Re^{ik\xi_1V_M(0)}}{(R+s)^2}\,\bar{f}(k\xi_2)+\frac{R^2e^{ik\xi_1^2V_M(0)}}{(R+s)^3}\,\bar{f}(k\xi_2\xi_1)\,\bar{f}(k\xi_2)+\left(\frac{R}{R+s}\right)^3\bar{W}(k\xi_1^3,s)\,\bar{f}(k\xi_2\xi_1^2)\,\bar{f}(k\xi_2\xi_1)\,\bar{f}(k\xi_2). \tag{84}$$

Continuing this procedure yields

$$\bar{W}(k,s)=\frac{e^{ikV_M(0)}}{R+s}+\sum_{n=1}^{\infty}\frac{R^n}{(R+s)^{n+1}}\,e^{ik\xi_1^nV_M(0)}\Pi_{i=1}^{n}\,\bar{f}(k\xi_1^{n-i}\xi_2). \tag{85}$$

Inverting to the time domain, using the inverse Laplace $s\to t$ transform yields Eq. (8). The solution Eq. (8) may be verified by substitution in Eq. (7).

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